Gaussian Fluctuation for the Number of Particles in Airy, Bessel, Sine and Other Determinantal Random Point Fields

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Abstract

We prove the Central Limit Theorem for the number of eigenvalues near the spectrum edge for hermitian ensembles of random matrices. To derive our results, we use a general theorem, essentially due to Costin and Lebowitz, concerning the Gaussian fluctuation of the number of particles in random point fields with determinantal correlation functions. As another corollary of Costin-Lebowitz Theorem we prove CLT for the empirical distribution function of the eigenvalues of random matrices from classical compact groups.

1 Introduction and Formulation of Results

Random hermitian matrices were introduced in mathematical physics by Wigner in the fifties ([Wig1], [Wig2]). The main motivation of pioneers

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in this field was to obtain a better understanding of the statistical behavior of energy levels of heavy nuclei. An archetypical example of random matrices is the Gaussian Unitary Ensemble (G.U.E.) which can be defined by the probability distribution on a space of n-dimensional hermitian matrices as

$$P(dA) = \text{const}_n \cdot e^{-2n \ \text{Trace} \ A^2} dA. \tag{1.1}$$

Here dA is the Lebesgue measure on n^2-parameters set

$$\{\text{Re} \ a_{ij}, 1 \leq i < j \leq n; \ \text{Im} \ a_{ij}, \ 1 \leq i < j \leq n, \ a_{ii}, \ 1 \leq i \leq n\} \tag{1.2}$$

and $\text{const}_n = (\pi n)^{-\frac{n^2}{2}} \cdot 2^{n(2n-1)/2}$ is a normalization constant. (1.1) implies that matrix entries (1.2) are independent Gaussian random variables $N(0, \frac{1+\delta_{ij}}{8n})$. It is well known that the G.U.E. is the only ensemble of hermitian random matrices (up to a trivial rescaling) that satisfies both of the following properties:

(1) probability distribution $P(dA)$ is invariant under unitary transformation

$$A \rightarrow U^{-1}AU, \ U \in U(n),$$

(2) matrix entries up from the diagonal are independent random variables (see [Me], Ch. 2).

The n eigenvalues, all real, of hermitian matrix A will be denoted by λ_1, $\lambda_2, \ldots, \lambda_n$. For the formulas for their joint distribution density $p_n(\lambda_1, \ldots, \lambda_n)$ and k-point correlation functions $\rho_{n,k}(\lambda_1, \ldots, \lambda_k)$ we refer to Mehta's book [Me]. One has

$$p_n(\lambda_1, \ldots, \lambda_n) = \text{const}'_n \cdot \prod_{1 \leq i < j \leq n} |\lambda_i - \lambda_j|^2 \cdot exp(-2n \cdot \sum_{i=1}^{n} \lambda_i^2) \tag{1.3}$$

$$\rho_{n,k}(\lambda_1, \ldots, \lambda_k) := \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(\lambda_1, \ldots, \lambda_n) d\lambda_{k+1} \ldots d\lambda_n$$
$$= \det(K_n(\lambda_i, \lambda_j))_{i,j=1}^{k}, \tag{1.4}$$

where $K_n(x, y)$ is a projection kernel,

$$K_n(x, y) = \sqrt{2n} \cdot \sum_{\ell=0}^{n-1} \psi_\ell(\sqrt{2n}x) \cdot \psi_\ell(\sqrt{2n} \cdot y) \tag{1.5}$$

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and

$$\psi_\ell(x) = \frac{(-1)^\ell}{\pi^{\frac{1}{4}} \cdot (2^\ell \cdot \ell!)^{\frac{1}{2}}} \cdot \exp\left(\frac{x^2}{2}\right) \cdot \frac{d^\ell}{dx^\ell} \exp(-x^2) \tag{1.6}$$

$\ell = 0, 1, \ldots,$ are Weber-Hermite functions. The global behavior of eigenvalues is governed by the celebrated semicircle law, which states that the empirical distribution function of the eigenvalues weakly converges to a non-random (Wigner) distribution:

$$\mathcal{F}_n(\lambda) = \frac{1}{n} \#\{\lambda_i \leq \lambda\} \xrightarrow[n \to \infty]{w} \mathcal{F}(\lambda) = \int_{-\infty}^{\lambda} \rho(x)dx \tag{1.7}$$

with probability one ([Wig1], [Wig2]) where the spectral density ρ is given by

$$\rho(t) = \begin{cases} \frac{2}{\pi}\sqrt{1 - t^2}, & |t| \leq 1 \\ 0, & |t| > 1. \end{cases} \tag{1.8}$$

To study the local behavior of eigenvalues near an arbitrary point in the spectrum $x \in [-1, 1]$, one has to consider rescaling

$$\lambda_i = x + \frac{y_i}{\rho_{n,1}(x)}, \quad i = 1, \ldots, k \tag{1.9}$$

and study the rescaled k-point correlation functions

$$R_{n.k}(y_1, \ldots, y_k) := (\rho_{n,1}(x))^{-k} \cdot \rho_{n,k}(\lambda_1, \ldots, \lambda_k). \tag{1.10}$$

The biggest interest is paid to the asymptotics of rescaled correlation functions when n goes to infinity. For G.U.E. the answer can be obtained from the Plancherel-Rotach asymptotic formulas for Hermite polynomials ([PR]):

$$\lim_{n \to \infty} R_{n,k}(y_1, \ldots, y_k) = \rho_k(y_1, \ldots, y_k) = \det(K(y_i, y_j))_{i,j=1}^k. \tag{1.11}$$

The K actually also depends on x but in a very simple way. It can be represented as

$$K(y, z) = \frac{\mathcal{A}(y) \cdot \mathcal{A}'(z) - \mathcal{A}(z)\mathcal{A}'(y)}{y - z}, \tag{1.12}$$

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where for all $|x| < 1$ the function $\mathcal{A}$ is just $\frac{\sin(\pi y)}{\pi}$, and for $x = \pm 1$ it is

$$\mathcal{A}i(\pm y) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 \pm yt\right) dt. \tag{1.13}$$

The function defined by (1.13) is known as the Airy function and the kernel (1.12)–(1.13) is known as the Airy kernel (see [Me], [TW1], [F]). The limiting correlation functions (1.11)-(1.13) determine a random point field on the real line, i.e., probability measure on the Borel σ-algebra of the space of locally finite configurations,

$$\Omega = \{\omega = (y_i)_{i\in\mathbb{Z}} : \forall T > 0 \ \#\{y_i : |y_i| < T\} < \infty\}. \tag{1.14}$$

The distribution of random point field is uniquely defined by the generating function

$$\psi(z_1, \dots, z_k; \ I_1, \dots, I_k) = E \prod_{j=1}^k z_j^{\nu_j},$$

where I_j, $j = 1, \dots, k$, are disjoint intervals on the real line, $\nu_j = \#\{y_i \in I_j\} = \#(I_j)$, the number of particles in I_j, and $k \in \mathbb{Z}_+^1$. It follows from the general theory of existence and uniqueness for random point fields ([L1], [L2]), that if $K(y, z)$ is locally bounded than determinantal correlation functions uniquelly determine random point field assuming that such randompoint fiels exists. The generating function $\psi(z_1, \dots, z_k)$ is given by Fredholm determinant of the intergal operator in $L^2(\mathbb{R}^1)$:

$$\psi(z_1, \dots, z_k) = \det(\delta(x - y) + \sum_{j=1}^k (z_j - 1) \cdot K(x, y) \cdot \chi_{I_j}(y)), \tag{1.15}$$

where χ_{I_j} is an indicator of I_j.

In particular these results are applicable to the Airy kernel (1.12)–(1.13). We shall call the corresponding random point field the Airy random point field. For one-level density formulas (1.11)–(1.13) produce

$$\rho_1(y) = -y \cdot (\mathcal{A}i)^2(y) + (\mathcal{A}i'(y))^2. \tag{1.16}$$

The asymptotic expansion of the Airy function is well known (see [Ol]). One can deduce from it

$$\rho_1(y) \sim \begin{cases} \frac{|y|^{\frac{1}{2}}}{\pi} - \frac{\cos(4 \cdot |y|^{\frac{3}{2}}/3)}{4\pi \cdot |y|} + \underline{0}(|y|^{-\frac{5}{2}}) & \text{as } y \to -\infty, \\ \frac{17}{96 \cdot \pi y^{\frac{1}{2}}} \cdot \exp(-4y^{\frac{3}{2}}/3) & \text{as } y \to +\infty. \end{cases} \tag{1.17}$$

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ρ_1 satisfies the third order differential equation

$$\rho_1'''(y) = -2\rho_1(y) + 4y \cdot \rho_1'(y). \qquad (1.18)$$

One can think about the one-point correlation function as a level density, since for any interval $I \subset \mathbb{R}^1$ we have $E\#(I) = \int_I \rho_1(y)dy$. It follows from (1.17) that $E\#((-T, +\infty))$ is finite for any T and $E\#((-T, +\infty)) \sim \frac{2T^{\frac{3}{2}}}{3\pi} + \underline{0}(1)$ when T goes to $+\infty$. The last formula means that $E\#((-T, +\infty)) - \frac{2T^{\frac{3}{2}}}{3\pi}$ stays bounded for large positive T. Let us denote $\nu_1(T) := \#\{y_i > -T\} = \#((-T, +\infty))$, $\nu_k(T) := \#((-kT, -(k-1)T]))$, $k = 2, 3, \ldots$. Theorem 1 establishes the Central Limit Theorem for $\nu_k(T)$.

Theorem 1 *The variance of $\nu_k(T)$ grows logarithmically*

$$\mathrm{Var}\ \nu_k(T) \sim \frac{11}{12\pi^2} \cdot \log T + \underline{0}(1),$$

and the sequence of normalized random variables $\frac{\nu_k(T) - E\nu_k(T)}{\sqrt{\mathrm{Var}\ \nu_k(T)}}$ converges in distribution to the centalized gaussian random sequence $\{\xi_k\}$ with the covariance function $E\xi_k\xi_l = \delta_{k,l} - 1/2\ \delta_{k,l+1} - 1/2\ \delta_{k,l-1}$.

Remark 1. The first result about Gaussian fluctuation of the number of particles in random matrix model was established by Costin and Lebowitz ([CL]) for the kernel $\frac{\sin \pi(x-y)}{\pi(x-y)}$. See §2 for a more detailed discussion.

Remark 2. Basor and Widom ([BaW]) recently proves the Central Limit Theorem for a large class of smooth linear statistics $\sum_{i=-\infty}^{+\infty} f(y_i/T)$ where f satisfies some decay and differentiality conditions. Similar results for smooth linear statistics in other random matrix ensembles were proven in [Sp],[DS], [Jo1], [SiSo1], [KKP], [SiSo2], [Ba], [BF], [BdMK], [Br]; see also [So1] for the results about global distribution of spacings.

Another class of random hermitian matrices, called Laguerre ensemble, was introduced by Bronk in [Br]. This one is the ensemble of positive $n \times n$ hermitian matrices. Any positive hermitian matrix H can be represented as $H = AA^*$, where A is some complex valued $n \times n$ matrix and A^* is its conjugate. The distribution on such matrices is defined as

$$P(dH) = \mathrm{const}_n'' \cdot \exp(-n \cdot \mathrm{Trace}\ A \cdot A^*) \cdot [\det(AA^*)]^\alpha dA, \qquad (1.19)$$

where $\alpha > -1$ and dA is Lebesgue measure on $2n^2$-dimensional space of complex matrices. The joint distribution of n (positive) eigenvalues of H is

given by

$$p_n(\lambda_1, \dots, \lambda_n) = \mathrm{const}_n''' \cdot \exp\left(-n \cdot \sum_{i=1}^n \lambda_i\right) \cdot \prod_{i=1}^n \lambda_i^\alpha \cdot \prod_{1 \le i < j \le n} (\lambda_i - \lambda_j)^2.$$

(1.20)

The Vandermonde factor in (1.20) implies that correlation functions still have the determinantal form (1.4) with the kernel

$$K_n(x, y) = n \cdot \sum_{\ell=0}^{n-1} \phi_\ell(nx) \cdot \phi_\ell(ny),$$

(1.21)

where the sequence $\{\phi_\ell(x)\}$ is obtained by orthonormalizing the sequence $\{x^k \cdot x^{\frac{\alpha}{2}} \cdot e^{-x/2}\}$ on $(0, +\infty)$. The limiting level density is supported on $[0, 1]$ and given by the equation

$$\rho_1(x) = \frac{1}{2\pi} x^{-\frac{1}{2}} \cdot (1 - x)^{\frac{1}{2}}.$$

(1.22)

(It is not surprising to see that (1.22) is the density of a square of the Wigner random variable!) Plancherel-Rotach type asymptotics for Laguerre polynomials ([E], [PR]) imply that by scaling the kernel $K_n(x, y)$ in the bulk of the spectrum, we obtain the sine kernel $\frac{\sin \pi(x-y)}{\pi(x-y)}$, and by scaling at $x = 1$ ("soft edge"), we obtain the Airy kernel. As we already know, the same kernels appear after rescaling in G.U.E. This feature, called universality of local correlations, has been established recently for a variety of ensembles of hermitian random matrices (see [PS],[BI],[DKMVZ],[Jo2],[So2],[BZ]). To scale the kernel at the "hard edge" $x = 0$, we need an asymptotic formula of Hilb's type (see [E]), which leads to

$$\lim_{n \to \infty} \frac{1}{4n} K_n\left(\frac{x}{4n}, \frac{y}{4n}\right) = \frac{J_\alpha(\sqrt{x}) \cdot \sqrt{y} \cdot J_\alpha'(\sqrt{y}) - \sqrt{x} J_\alpha'(x) \cdot J_\alpha(y)}{2(x - y)}, \quad (1.23)$$

where J_α is the Bessel function of order α ([F], [TW2], [Ba]). The kernel (1.23) is also known to appear at hard edges in the Jacobi ensemble ([NW]). For a quick reference, we note that in the Jacobi case, a sequence $\{\phi_\ell(x)\}$ from (1.21) is obtained by orthonormalizing $\{x^k(1 - x)^{\frac{\alpha}{2}}(1 + x)^{\frac{\beta}{2}}\}$. The random point field on $[0, +\infty]$ with the determinantal correlation functions defined by (1.23) will be referred to as the Bessel random point field. There

is a general belief among people working in random matrix theory that in the same way as the sine kernel appears to be a universal limit in the bulk of the spectrum for random hermitian matrices, Airy and Bessel kernels are universal limits at the soft and hard edge of the spectrum. The next theorem establishes the CLT for $\nu_k(T) = \#(((k-1)T, kT]), k = 1, 2, \ldots$.

Theorem 2 *Let $\nu(T)$ be the number of particles in $(0, T)$ for the Bessel random point field. Then*

$$E \ \nu_k(T) \sim \frac{1}{\pi} T^{\frac{1}{2}} (k^{1/2} - (k-1)^{1/2} + \underline{0}(1),$$

$$\mathrm{Var} \ \nu_k(T) \sim \frac{1}{4\pi^2} \log T + \underline{0}(1),$$

and the sequence of the the normalized random variable $\frac{\nu_k(T) - E \ \nu_k(T)}{\sqrt{Var \ \nu_k(T)}}$ converges in distribution to the gaussian random sequence from the Theorem 1.

Theorems 1 and 2, as well as similar results for the random fields arising from the classical compact groups (see [So1]) are the corollaries of the general result about determinantal random point fields, which is essentially due to Costin and Lebowitz. Recently a number of discrete determinantal random point fields appeared in two-dimensional growth models ([Jo3],[Jo5]), asymptotics of Plancherel measures on symmetric groups and the representation theory of the infinite symmetric group ([BO1], [BO2],[BOO],[Jo4],[Ok1],[Ok2]). If one can show the infinite growth of the variance of the number of particles in these models (the goal which may be probably attainable since the asymptotics of the discrete orthogonal polynomials arising in some of these problems are known) the Costin- Lebowitz theorem should work there as well.

The rest of the paper is organized as follows. We discuss the general (Costin-Lebowitz) theorem in §2. Theorems 1 and 2 will be proven in §3 and §4. In the Bessel case, we will see that the kernel $\frac{\sin \pi(x-y)}{\pi(x-y)} \pm \frac{\sin \pi(x+y)}{\pi(x+y)}$ naturally appears in our considerations. We recall in §4 that the sine kernel also appears in the limiting distribution of eigenvalues in unitary group and the even and odd sine kernels appear in the distribution of eigenvalues in orthogonal and symplectic groups and then prove Theorems 3-6 the Gaussian fluctuation for the number of eigenvalues in these models in Theorem 3-6.

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2 The Central Limit Theorem for Determinantal Random Point Fields

Let $\{\mathcal{P}_t\}_{t\in\mathbb{R}^1_+}$ be a family of random point fields on the real line such that their correlation functions have determinantal form at the r.h.s. of (1.11) with kernels $K_t(y, z)$, and $\{I_t\}_{t\in\mathbb{R}^1_+}$ a set of intervals. We denote by A_t an integral operator on I_t with the kernel $K_t(y, z)$, $A_t : L^2(I_t) \to L^2(I_t)$, by ν_t the number of particles in I_t, $\nu_t = \#(I_t)$, and by E_t, Var_t the mathematical expectation and variance with respect to the probability distribution of the random field $\mathcal{P}_t$. In many applications the random point field $\mathcal{P}_t$, and therefore the kernel K_t will be the same for all t. In such situations the interval I_t will be expanding.

Theorem (O. Costin, J. Lebowitz) *Let $A_t = K_t \cdot \chi_{I_t}$ be a family of trace class operators associated with determinantal random point fields $\{\mathcal{P}_t\}$ such that $\mathrm{Var}_t\, \nu_t = \mathrm{Trace}(A_t - A_t^2)$ goes to infinity as $t \to +\infty$. Then the distribution of the normalized random variable $\frac{\nu_t - E_t \nu_t}{\sqrt{Var_t\ \nu_t}}$ with respect to the random point field $\mathcal{P}_t$ weakly converges to the normal law $N(0, 1)$.*

Remark 3. The result has been proven by Costin and Lebowitz when $K_t(x, y) = \frac{\sin \pi(x-y)}{\pi(x-y)}$ for any t and $|I_t| \underset{t\to\infty}{\longrightarrow} \infty$ (see [CL]). The original paper contains a remark, due to Widom, that the result holds for more general kernels.
Remark 4. There is a general result that a (locally) trace class operator K defines a determinantal random point field iff $0 \le K \le 1$ (see [So4] or, for a slightly weaker version, [Ma]).

The idea of the proof is very clear and consists of two parts. Let us denote the ℓth cumulant of ν_t by $C_\ell(\nu_t)$. We remind that by definition

$$\sum_{\ell=1}^{\infty} C_\ell (iz)^\ell/\ell! = \log\left(E_t \exp(iz\nu_t)\right).$$

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Lemma 1 *The following recursive relation holds for any $\ell \geq 2$:*

$$C_\ell(\nu_t) = (-1)^\ell \cdot (\ell - 1)! \mathrm{Trace}(A_t - A_t^\ell) + \sum_{s=2}^{\ell-1} \alpha_{s\ell} C_s(\nu_t), \qquad (2.1)$$

where $\alpha_{s\ell}$, $2 \leq s \leq \ell - 1$, are some combinatorial coefficients (irrelevant for our purposes).

The proof can be found in [CL] or [So1], §2; (of course one has to replace everywhere $\frac{\sin \pi(x-y)}{\pi(x-y)}$ by $K_t(x, y)$). For the convinience of the reader we sketch the main ideas here. We start by introducing the Ursell (cluster) functions :

$$r_1(x_1) = \rho(x_1), \quad r_2(x_1, x_2) = \rho_2(x_1, x_2) - \rho_1(x_1)\rho(x_2),$$

and, in general,

$$r_k(x_1, \ldots, x_k) = \sum_{m=1}^{k} \sum_{G} (-1)^{m-1}(m-1)! \prod_{j=1}^{m} r_{G_j}(\bar{x}(G_j)) \qquad (2.2)$$

where G is a partition of indices $\{1, 2, \ldots, k\}$ into m subgroups $G_1, \ldots G_m$, and $\bar{x}(G_j)$ stands for the collection of x_i with indices in G_j .
It appears that the integral of k-point Ursell function $r_k(x_1, \ldots, x_k)$ over k-dimensional cube $I_t \times \ldots I_t$ is equal to the linear combination of $C_j(\nu_t)$, $j = 1, \ldots k$.. Namely, let us denote

$$T_k(\nu_t) = \int_{I_t} \ldots \int_{I_t} r_k(x_1, \ldots, x_k) dx_1 \ldots dx_k$$

Then

$$\sum_{k}^{\infty} C_k(iz)^k/k! = \sum_{k=1}^{\infty} (\exp(z) - 1)^k T_k(\nu_t)/k! \qquad (2.3)$$

Taking into account that for the determinantal random point fields

$$T_k(\nu_t) = (-1)^k \cdot (k - 1)! Trace(A_t)^k,$$

the last two equations imply (2.1). The next lemma allows us to estimate Trace $(A_t - A_t^\ell)$.

Lemma 2 $0 \leq \text{Trace}(A_t - A_t^\ell) \leq (\ell - 1) \cdot \text{Trace } (A_t - A_t^2).$

The proof is elementary: $0 \leq \text{Trace } (A_t - A_t^\ell) = \sum_{j=1}^{\ell-1} \text{Trace } (A_t^j - A_t^{j+1}) \leq \sum_{j=1}^{\ell-1} \|A_t^{j-1}\| \cdot \text{Trace } (A_t - A_t^2) \leq (\ell - 1) \cdot \text{Trace } (A_t - A_t^2).$ □

As a corollary of the lemmas we have $C_\ell(\nu_t) = \underline{0}(C_2(\nu_t))$ for any $\ell \geq 2$. Since $C_2(\nu_t) = \text{Trace } (A_t - A_t^2) \xrightarrow[t\to\infty]{} +\infty$, we conclude that for $\ell > 2$, $C_\ell\left(\frac{\nu_t - E\nu_t}{\sqrt{var_t\nu_t}}\right) = \frac{C_\ell(\nu_t)}{((C_2(\nu_t))^\ell/2} \xrightarrow[t\to\infty]{} 0.$

At the same time the first two cumulants of the normalized random variable are 0 and 1, respectively. The convergence of cumulants implies the convergence of moments to the moments of $N(0,1)$. The theorem is proven. □

To generalize the Costin-Lebowitz theorem to the case of several intervals we consider $I_t^{(m)}$, $m + 1, \ldots, s$, disjoint intervals of the real line, and define $\nu_t^{(m)} = \#(I_t^{(m)})$. The equation

$$\sum C_{k_1,\ldots,k_s}(iz_1)^{k_1}/k_1! \cdots (iz_s)^{k_s}/k_s! = \log\left(E_t \exp(i(z_1\nu_t^{(1)} + \ldots + z_s\nu_t^{(s)}))\right) \tag{2.4}$$

defines the joint cumulants of $\nu_t^{(m)}$'s.

Proposition 1 $C_{k_1,\ldots,k_s}\left(\nu_t^{(1)}, \ldots, \nu_t(s)\right)$ *is equal to the linear combination of the traces*

$$\text{Trace} K_t \cdot \chi_{I_t}^{(\cdots)} \cdot K_t \cdot \chi_{I_t}^{(\cdots)} \ldots K_t \cdot \chi_{I_t}^{(\cdots)}$$

with some combinatorial coefficients (irrelevant for our purposes), such that for any k_j that is greater than zero at least one indicator in each term of the linear combination is the indicator of $I_t^{(j)}$.

The proof immedeately follows from the analogue of (2.4) for the case of a several intervals.

In the next section we will apply theses results to prove Theorem 1.

3   Proof of Theorem 1

For the most part of the section we will study the case of one interval $(-T, +\infty)$. We start by recalling the asymptotic expansion of Airy function for large positive and negative y (see [Ol]).

$$\mathcal{A}_i(|y|) \sim \frac{e^{-\pi z}}{2\pi^{\frac{1}{2}} \cdot |y|^{\frac{1}{4}}} \cdot \sum_{s=0}^\infty (-1)^s \cdot \frac{u_s}{z^s}; \tag{3.1}$$

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$$\mathcal{A}'_i(|y|) \sim \frac{|y|^{\frac{1}{4}} \cdot e^{-\pi z}}{2\pi^{\frac{1}{2}}} \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{v_s}{z^s} \tag{3.2}$$

$$\mathcal{A}_i(-|y|) \sim \frac{1}{\pi^{\frac{1}{2}} \cdot |y|^{\frac{1}{4}}} \cdot \left\{ \cos\left(\pi z + \frac{\pi}{4}\right) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{u_{2s}}{z^{2s}} \right.$$
$$\left. + \sin(\pi z + \frac{\pi}{4}) \cdot \sum_{s=0}^{\infty} (-1)^s \cdot \frac{u_{2s+1}}{z^{2s+1}} \right\}, \tag{3.3}$$

$$\mathcal{A}'_i(-|y|) \sim \frac{|y|^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} \cdot \left\{ \sin\left(\pi z + \frac{\pi}{4}\right) \cdot \sum_{s=0}^{\infty} (-1)^{s+1} \cdot \frac{v_{2s}}{z^{2s}} \right.$$
$$\left. - \cos(\pi z + \frac{\pi}{4}) \cdot \sum_{s=0}^{\infty} (-1)^{s+1} \cdot \frac{v_{2s+1}}{z^{2s+1}} \right\}, \tag{3.4}$$

where $z = \frac{2}{3\pi} y \cdot |y|^{\frac{1}{2}}$; $u_0 = v_0 = 1$, and $u_s = \frac{(2s+1)\cdot(2s+3)\cdot\ldots\cdot(6s-1)}{(216\cdot\pi)^2\cdot s!}$, $v_s = -\frac{6s+1}{6s-1} u_s$, $s \geq 1$. In particular, as a consequence of (3.1)–(3.4) one has (1.17). It follows from (3.1)–(3.4) together with the boundedness of $\mathcal{A}_i(y)$, $\mathcal{A}'_i(y)$ on any compact set that for any fixed $a \in \mathbb{R}^1$ all moments of $\#((a, +\infty))$ are finite. Therefore it is enough to establish the CLT for $\#((-T, a))$. We choose $a = -(\frac{3\pi}{2})^{\frac{2}{3}}$ ($y = -(\frac{3\pi}{2})^{\frac{2}{3}}$ corresponds to $z = -1$). We are going to show that the conditions of the theorem from §2 are satisfied by $K \cdot \chi_{(-T,a)}$, where as above, this notation is reserved for the integral operator with the kernel $K(x, y) \cdot \chi_{(-T,a)}(y)$.

Lemma 3 $0 \leq K \cdot \chi_{(-T,a)} \leq 1$ and $K \cdot \chi_{(-T,a)}$ is trace class. $\qquad \square$

The kernel $K(y_1, y_2) = \frac{\mathcal{A}_i(y_1)\cdot\mathcal{A})i'(y_2) - \mathcal{A}'_i(y_1)\cdot\mathcal{A}_i(y_2)}{y_1 - y_2}$ was obtained from $K_n(x_1, x_2) = \sqrt{2n} \cdot \sum_{\ell=0}^{n-1} \psi_\ell(\sqrt{2n} x_1) \cdot \psi_\ell(\sqrt{2n} \cdot x_2)$ after rescaling $x_i = 1 + \frac{y_i}{2n^{\frac{2}{3}}}$, $i = 1, 2$, and taking the limit $n \to \infty$. The convergence is uniform on compact sets. As a projection operation K_n satisfies $0 \leq K_n \leq 1$. We immediately conclude that $K \cdot \chi_{(-T,a)}$ satisfies the same inequalities and since the kernel is continuous and non-negative definite the operator is trace class (see e.g [GK] or [RS], section XI.4). Now the main step of the proof consists of

Proposition 2.

$$\mathrm{Var}\left(\#\left(-T, -\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}\right)\right) \sim \frac{11}{12\pi^2} \log T + \underline{0}(1).$$

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Proof. We introduce the change of variables

$$z_i = \frac{2}{3\pi} y_i \cdot |y_i|^{\frac{1}{2}} \tag{3.5}$$

and agree to use the notations $Q(z_1, z_2)$, $q_k(z_1, \ldots, z_k)$, $k = 1, 2, \ldots$, for the kernel and k-point correlation function of the new random point field obtained by (3.5). It follows from (1.17) that $q_1(z) \sim 1 + \frac{\cos 2\pi z}{6\pi z} + \underline{0}(z^{-2})$ for $z \to -\infty$, so we see that the configuration (z_i) is equally spaced at $-\infty$.

The kernel $Q(z_1, z_2)$ is defined by

$$Q(z_1, z_2) = \frac{\pi}{|y_1|^{\frac{1}{4}} \cdot |y_2|^{\frac{1}{4}}} \cdot K(y_1, y_2)$$

$$= \frac{\pi}{|y_1|^{\frac{1}{4}} \cdot |y_2|^{\frac{1}{4}}} \cdot \frac{\mathcal{A}_i(y_1) \cdot \mathcal{A}_i'(y_2) - \mathcal{A}_i'(y_1) \cdot \mathcal{A}_i(y_2)}{y_1 - y_2}.$$

Formulas (3.3)–(3.4) allow us to represent Q as the sum of six kernels $Q^{(i)}$, $i = 1, \ldots, 6$, with the known asymptotic expansion: $Q(z_1, z_2) = \sum_{i=1}^{6} Q^{(i)}(z_1, z_2)$, where

$$Q^{(1)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot \sin \pi(z_1 - z_2) \cdot \left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \right.$$

$$\left. \cdot u_{2m} \cdot v_{2n} \cdot \left(z_1^{-2m-\frac{1}{3}} \cdot z_2^{-2n} + z_1^{-2n} \cdot z_2^{-2m-\frac{1}{3}} \right) \right\}; \tag{3.6}$$

$$Q^{(2)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot \cos \pi(z_1 + z_2) \cdot$$

$$\left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \cdot u_{2m} \cdot v_{2n} \cdot \left(z_1^{-2n} \cdot z_2^{-2m-\frac{1}{3}} - z_1^{-2m-\frac{1}{3}} \cdot z_2^{-2n} \right) \right\}; \tag{3.7}$$

$$Q^{(3)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot 2 \cos \left(\pi z_1 + \frac{\pi}{4} \right) \cdot \cos \left(\pi z_2 + \frac{\pi}{4} \right) \cdot$$

$$\left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n+1} \cdot u_{2m} \cdot v_{2n+1} \cdot \left(z_1^{-2m-\frac{1}{3}} \cdot z_2^{-2n-1} - z_1^{-2n-1} \cdot z_2^{-2m-\frac{1}{3}} \right) \right\}; \tag{3.8}$$

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$$Q^{(4)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot 2\sin\left(\pi z_1 + \frac{\pi}{4}\right) \cdot \sin\left(\pi z_2 + \frac{\pi}{4}\right) \cdot$$
$$\left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \cdot u_{2m+1} \cdot v_{2n} \cdot \left(z_1^{-2m-\frac{4}{3}} \cdot z_2^{-2n} - z_1^{-2n} \cdot z_2^{-2m-\frac{4}{3}} \right) \right\};$$

$$(3.9)$$

$$Q^{(5)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot \sin \pi(z_1 - z_2) \cdot$$
$$\left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n+1} \cdot u_{2m+1} \cdot v_{2n+1} \cdot \left(z_1^{-2m-\frac{4}{3}} \cdot z_2^{-2n-1} + z_1^{-2n-1} \cdot z_2^{-2m-\frac{4}{3}} \right) \right\};$$

$$(3.10)$$

$$Q^{(6)}(z_1, z_2) \sim \frac{1}{3\pi} \cdot \frac{1}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot \cos \pi(z_1 + z_2) \cdot$$
$$\left\{ \sum_{m,n=0}^{\infty} (-1)^{m+n} \cdot u_{2m+1} \cdot v_{2n+1} \cdot \left(z_1^{-2n-1} \cdot z_2^{-2m-\frac{4}{3}} - z_1^{-2m-\frac{4}{3}} \cdot z_2^{-2n-1} \right) \right\}.$$

$$(3.11)$$

We denote by $Q^{(i)}_{m,n}(z_1, z_2)$ the $(m,n)^{th}$ term in the asymptotic expansion of $Q^{(i)}(z_1, z_2)$. Then

$$Q^{(1)}_{0,0}(z_1, z_2) = \frac{\sin \pi(z_1 - z_2)}{z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}} \cdot \frac{1}{3\pi} \cdot \left(z_1^{-\frac{1}{3}} + z_2^{-\frac{1}{3}} \right)$$
$$= \frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} \cdot \frac{z_1^{\frac{2}{3}} + z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}} + z_2^{\frac{2}{3}}}{3 \cdot z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}}}.$$

We note that near the diagonal $Q^{(1)}_{0,0}$ is essentially a sine kernel. We also will need

$$Q^{(2)}_{0,0}(z_1, z_2) = \frac{\cos \pi(z_1 + z_2)}{\pi(z_1 + z_2)} \cdot \frac{z_1^{\frac{2}{3}} - z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}} + z_2^{\frac{2}{3}}}{3 \cdot z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}}}.$$

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Let us define $S(z_1, z_2) = Q_{0,0}^{(1)}(z_1, z_2) + Q_{0,0}^{(2)}(z_1, z_2)$, $U(z_1, z_2) = Q(z_1, z_2) - S(z_1, z_2)$.

Lemma 4.

$$\int_{-L}^{-1} \int_{-L}^{-1} \left(Q_{0,0}^{(1)}(z_1, z_2) \right)^2 dz_1 dz_2 = L - \frac{2}{3\pi^2} \log L + \underline{0}(1). \qquad (3.12)$$

Proof. The integral can be written as

$$\frac{1}{9} \int_1^L \int_1^L \left(\frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} \right)^2 \cdot \left(\frac{z_1^{\frac{2}{3}} + z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}} + z_2^{\frac{2}{3}}}{z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}}} \right)^2 dz_1 dz_2$$

$$= \frac{2}{9} \cdot \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot \int_1^{L-u} \left[\left(\frac{z+u}{z} \right)^{\frac{1}{3}} + 1 + \left(\frac{z}{z+u} \right)^{\frac{1}{3}} \right]^2 dz \, du.$$

We represent the inner integral as

$$\int_1^{L-u} \left[\left(\frac{z+u}{z} \right)^{\frac{2}{3}} + 2 \cdot \left(\frac{z+u}{z} \right)^{\frac{1}{3}} + \left(\frac{z}{z+u} \right)^{\frac{2}{3}} + 2 \cdot \left(\frac{z}{z+u} \right)^{\frac{1}{3}} + 3 \right] dz$$

$$= I_1(u) + I_2(u) + 3 \cdot (L - u - 1),$$

$$(3.13)$$

with

$$I_1(u) = \int_1^{L-u} \left(\frac{z+u}{z} \right)^{\frac{2}{3}} + 2 \cdot \left(\frac{z+u}{z} \right)^{\frac{1}{3}} dz,$$

$$I_2(u) = \int_1^{L-u} \left(\frac{z}{z+u} \right)^{\frac{2}{3}} + 2 \cdot \left(\frac{z}{z+u} \right)^{\frac{1}{3}} dz.$$

To calculate $I_1(u)$ we introduce the change of variables $t = (\frac{z+u}{z})^{\frac{1}{3}}$. Then

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$$I_1(u) = \int_{(\frac{L}{L-u})^{\frac{1}{3}}}^{(1+u)^{\frac{1}{3}}} (t^2 + 2t) \cdot \left(\frac{u}{1 - t^3} \right)' dt = \left((1+u)^{\frac{2}{3}} + 2 \cdot (1+u)^{\frac{1}{3}} \right)$$

$$\cdot (-1) - \left(\left(\frac{L}{L-u} \right)^{\frac{2}{3}} + 2 \cdot \left(\frac{L}{L-u} \right)^{\frac{1}{3}} \right) \cdot (-L + u)$$

$$+ u \int_{(\frac{L}{L-u})^{\frac{1}{3}}}^{(1+u)^{\frac{1}{3}}} (2t + 2) \cdot \frac{1}{t^3 - 1} dt.$$

(3.14)

We have

$$u \cdot \int_{(\frac{l}{L-u})^{\frac{1}{3}}}^{(1+u)^{\frac{1}{3}}} (2t + 2) \cdot \frac{1}{t^3 - 1} dt = u \cdot \int_{(\frac{L}{L-u})^{\frac{1}{3}}}^{(1+u)^{\frac{1}{3}}} \frac{4}{3} \cdot \left(\frac{1}{t - 1} - \frac{t + \frac{1}{2}}{t^2 + t + 1} \right) dt$$

$$= \frac{4u}{3} \cdot \log \left[(1+u)^{\frac{1}{3}} - 1 \right] - \frac{4u}{3} \log \left[\left(\frac{L}{L-u} \right)^{\frac{1}{3}} - 1 \right]$$

$$- \frac{2u}{3} \log \left[(1+u)^{\frac{2}{3}} + (1+u)^{\frac{1}{3}} + 1 \right]$$

$$+ \frac{2u}{3} \log \left[\left(\frac{L}{L-u} \right)^{\frac{2}{3}} + \left(\frac{L}{L-u} \right)^{\frac{1}{3}} + 1 \right] \quad \text{and}$$

(3.15)

$$I_1(u) = -(1+u)^{\frac{2}{3}} - 2(1+u)^{\frac{1}{3}} + L \cdot \left(1 - \frac{u}{L} \right)^{\frac{1}{3}} + 2L \cdot \left(1 - \frac{u}{L} \right)^{\frac{2}{3}}$$

$$+ \frac{4u}{3} \log \left[(1+u)^{\frac{1}{3}} - 1 \right] - \frac{4u}{3} \log \left[\left(\frac{L}{L-u} \right)^{\frac{1}{3}} - 1 \right]$$

$$+ \frac{2u}{3} \log \left[\left(\frac{L}{L-u} \right)^{\frac{2}{3}} + \left(\frac{L}{L-u} \right)^{\frac{1}{3}} + 1 \right]$$

$$- \frac{2u}{3} \log \left[(1+u)^{\frac{2}{3}} + (1+u)^{\frac{1}{3}} + 1 \right].$$

(3.16)

In a similar way in order to calculate

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$$I_2(u) = \int_{1+u}^{L} \left(\frac{z-u}{z}\right)^{\frac{2}{3}} + 2 \cdot \left(\frac{z-u}{z}\right)^{\frac{1}{3}} dz,$$

we consider the change of variables $t = \left(\frac{z-u}{z}\right)^{\frac{1}{3}}$. Then

$$I_2(u) = \int_{(1+u)^{-\frac{1}{3}}}^{\left(\frac{L}{L-u}\right)^{-\frac{1}{3}}} (t^2 + 2t) \cdot \left(\frac{u}{1-t^3}\right)' dt = \left(\left(\frac{L-u}{L}\right)^{\frac{2}{3}} + \right.$$

$$\left. 2 \cdot \left(\frac{L-u}{L}\right)^{\frac{1}{3}}\right) \cdot L - \left((1+u)^{-\frac{2}{3}} + 2 \cdot (1+u)^{-\frac{1}{3}}\right) \cdot (1+u) +$$

$$u \cdot \int_{(1+u)^{-\frac{1}{3}}}^{\left(\frac{L}{L-u}\right)^{-\frac{1}{3}}} (2t+2) \cdot \frac{1}{t^3-1} dt = -(1+u)^{\frac{1}{3}} - 2(1+u)^{\frac{2}{3}}$$

$$+ L \cdot \left(1 - \frac{u}{L}\right)^{\frac{2}{3}} + 2L \cdot \left(1 - \frac{u}{L}\right)^{\frac{1}{3}} + \frac{4}{3} u \cdot \log\left[1 - \left(\frac{L-u}{L}\right)^{\frac{1}{3}}\right]$$

$$- \frac{4}{3} u \cdot \log\left[1 - (1+u)^{-\frac{1}{3}}\right] - \frac{2}{3} u \cdot \log\left[\left(\frac{L-u}{L}\right)^{\frac{2}{3}}\right.$$

$$+ \left(\frac{L-u}{L}\right)^{\frac{1}{3}} + 1\right] + \frac{2}{3} u \cdot \log\left[(1+u)^{-\frac{2}{3}} + (1+u)^{-\frac{1}{3}} + 1\right].$$

$$(3.17)$$

It follows from (3.13) that

$$\int_{-L}^{-1} \int_{-L}^{-1} \left(Q_{0,0}^{(1)}(z_1, z_2)\right)^2 dz_1 dz_2 =$$

$$\frac{2}{9} \cdot \int_{0}^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot (I_1(u) + I_2(u) + 3L - 3u - 3) du.$$

$$(3.18)$$

We note that

$$\int_{0}^{L-1} \left(\frac{\sin \pi u}{\pi u}\right)^2 \cdot L du = \frac{L}{2} + \underline{0}(1); \qquad (3.19)$$

$$\int_{0}^{L-1} \frac{(\sin \pi u)^2}{\pi u} du = \frac{1}{2\pi^2} \log L + \underline{0}(1); \qquad (3.20)$$

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$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot L \cdot \left(1 - \frac{u}{L} \right)^{\frac{1}{3}} du = L \cdot \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2$$

$$\cdot \left(1 - \frac{1}{3} \frac{u}{L} + \underline{0} \left(\frac{u^2}{L^2} \right) \right) du = \frac{L}{2} - \frac{1}{6\pi^2} \log L + \underline{0}(1); \qquad (3.21)$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot L \cdot \left(1 - \frac{u}{L} \right)^{\frac{2}{3}} du = \frac{L}{2} - \frac{1}{3\pi^2} \log L + \underline{0}(1). \qquad (3.22)$$

The combined contribution of all other terms to (3.18) is $\underline{0}(1)$. Indeed,

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot (1 + u)^{\frac{2}{3}} = \underline{0}(1);$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot (1 + u)^{\frac{1}{3}} = \underline{0}(1);$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot \frac{4u}{3} \log \left[(1 + u)^{\frac{1}{3}} - 1 \right] du - \int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2$$

$$\cdot \frac{2u}{3} \cdot \log \left[(1 + u)^{\frac{2}{3}} + (1 + u)^{\frac{1}{3}} + 1 \right] du =$$

$$\frac{2}{3\pi^2} \int_0^{L-1} \frac{\sin^2 \pi u}{\pi^2 u} \cdot \log \left[\frac{\left((1 + u)^{\frac{1}{3}} - 1 \right)^2}{(1 + u)^{\frac{2}{3}} + (1 + u)^{\frac{1}{3}} + 1} \right] du =$$

$$\frac{2u}{3\pi^2} \int_0^{L-1} \frac{\sin^2 \pi u}{u} \cdot \underline{0} \left(u^{-\frac{1}{3}} \right) du = \underline{0}(1);$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot \frac{4u}{3} \log \left[1 - (1 + u)^{-\frac{1}{3}} \right] du = \underline{0}(1);$$

$$\int_0^{L-1} \left(\frac{\sin \pi u}{\pi u} \right)^2 \cdot \frac{2u}{3} \log \left[(1 + u)^{-\frac{2}{3}} + (1 + u)^{-\frac{1}{3}} + 1 \right] du = \underline{0}(1).$$

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The last expression to consider is

$$-\int_0^{L-1}\left(\frac{\sin\pi u}{\pi u}\right)^2\cdot\frac{4}{3}u\cdot\log\left[\left(\frac{L}{L-u}\right)^{\frac{1}{3}}-1\right]du+\int_0^{L-1}\left(\frac{\sin\pi u}{\pi u}\right)^2$$

$$\cdot\frac{2}{3}u\cdot\log\left[\left(\frac{L}{L-u}\right)^{\frac{2}{3}}+\left(\frac{L}{L-u}\right)^{\frac{1}{3}}+1\right]du+\int_0^{L-1}\left(\frac{\sin\pi u}{\pi u}\right)^2\cdot\frac{4}{3}u$$

$$\cdot\log\left[1-\left(\frac{L-u}{L}\right)^{\frac{1}{3}}\right]du-\int_0^{L-1}\left(\frac{\sin\pi u}{\pi u}\right)^2\cdot\frac{2}{3}u\cdot\log\left[\left(\frac{L-u}{L}\right)^{\frac{2}{3}}\right.$$

$$\left.+\left(\frac{L-u}{L}\right)^{\frac{1}{3}}+1\right]du=\frac{2}{3\pi^2}\int_0^{L-1}\frac{\sin^2\pi u}{u}\cdot\log\left[\frac{\left(1-\left(\frac{L-u}{L}\right)^{\frac{1}{3}}\right)}{\left(\left(\frac{L}{L-u}\right)^{\frac{1}{3}}-1\right)}\right.$$

$$\times\frac{\left(\left(\frac{L}{L-u}\right)^{\frac{2}{3}}+\left(\frac{L}{L-u}\right)^{\frac{1}{3}}+1\right)}{\left(\left(\frac{L-u}{L}\right)^{\frac{2}{3}}+\left(\frac{L-u}{L}\right)^{\frac{1}{3}}+1\right)}\right]du=\frac{2}{3\pi^2}\int_0^{L-1}\frac{\sin^2\pi u}{u}\cdot\log\left(\frac{L}{L-u}\right)du$$

$$=\frac{1}{3\pi^2}\cdot\int_1^{L-1}\frac{1}{u}\cdot\left(-\log(1-\frac{u}{L})\right)du+\underline{0}(1)=\sqrt{\frac{1}{3\pi^2}}\cdot\int_1^{L-1}\frac{1}{u}\cdot\sum_{k=1}^\infty\frac{1}{k}\cdot$$

$$\cdot\left(\frac{u}{L}\right)^k du+\underline{0}(1)=\frac{1}{3\pi^2}\cdot\sum_{k=1}^\infty\frac{1}{k^2}+\underline{0}(1)=\underline{0}(1).$$

Combining all the above integrals and looking specifically for the contributions from (3.19)–(3.22) we obtain

$$\int_{-L}^{-1}\int_{-L}^{-1}\left(Q_{0,0}^{(1)}(z_1,z_2)\right)^2 dz_1dz_2=\frac{2}{9}\left(3\cdot\frac{L}{2}-3\cdot\frac{1}{2\pi^2}\log L+\frac{L}{2}-\frac{1}{6\pi^2}\cdot\right.$$

$$\log L+2\cdot\frac{L}{2}-2\cdot\frac{1}{3\pi^2}\cdot\log L+\frac{L}{2}-\frac{1}{3\pi^2}\log L+2\cdot\frac{L}{2}$$

$$\left.-2\cdot\frac{1}{6\pi^2}\log L+\underline{0}(1)\right)=L-\frac{2}{3\pi^2}\log L+\underline{0}(1).$$

$\square$

In the next lemma we evaluate integrals involving $Q_{0,0}^{(2)}$.

Lemma 5

a) $\int_{-L}^{-1} \int_{-L}^{-1} \left(Q_{0,0}^{(2)}(z_1, z_2) \right)^2 dz_1 dz_2 = \frac{1}{18\pi^2} \log L + \underline{0}(1)$.

b) $\int_{-L}^{-1} \int_{-L}^{-1} Q_{0,0}^{(1)}(z_1, z_2) \cdot Q_{0,0}^{(2)}(z_1, z_2) dz_1 dz_2 = \underline{0}(1)$.

Proof. The integral in part a) can be written as

$$\frac{1}{9} \int_1^L \int_1^L \left(\frac{\cos \pi(z_1 + z_2)}{\pi(z_1 + z_2)} \right)^2 \cdot \left(\frac{z_1^{\frac{2}{3}} - z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}} + z_2^{\frac{2}{3}}}{z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}}} \right)^2 dz_1 dz_2$$

$$= \frac{1}{9} \int_2^{2L} \left(\frac{\cos \pi u}{\pi u} \right)^2 \cdot \int_1^u \left[\left(\frac{z}{u-z} \right)^{\frac{1}{3}} - 1 + \left(\frac{u-z}{z} \right)^{\frac{1}{3}} \right]^2 dz\, du.$$

We denote the inner integral by

$$I_3(u) := \int_1^u \left[\left(\frac{z}{u-z} \right)^{\frac{2}{3}} - 2 \left(\frac{z}{u-z} \right)^{\frac{1}{3}} + \left(\frac{u-z}{z} \right)^{\frac{2}{3}} - 2 \left(\frac{u-z}{z} \right)^{\frac{1}{3}} + 1 \right] dz$$

$$= 2 \cdot \int_1^u \left(\frac{u-z}{z} \right)^{\frac{2}{3}} - 2 \left(\frac{u-z}{z} \right)^{\frac{1}{3}} dz + (u-1)$$

$$= 2 \cdot \int_{(u-1)^{\frac{1}{3}}}^0 (t^2 - 2t) \cdot \left(\frac{u}{t^3 + 1} \right)' dt + (u-1)$$

$$= 6u \cdot \int_0^{(u-1)^{\frac{1}{3}}} \frac{(t^2 - 2t) \cdot t^2}{(t^3 + 1)^2} dt + (u-1)$$

$$= u \left(1 + 6 \int_0^\infty \frac{t^4 - 2t^3}{(t^3 + 1)^2} dt \right) + \underline{0}(u^{\frac{2}{3}}).$$

Taking into account $\int_0^\infty \frac{t^4 - 2t^3}{(t^3 + 1)^2} dt = 0$, we obtain $I_3(u) = u + \underline{0}(u^{\frac{2}{3}})$, and

$$\frac{1}{9} \int_2^{2L} \left(\frac{\cos \pi u}{\pi u} \right)^2 \cdot (u + \underline{0}(u^{\frac{2}{3}})) du = \frac{1}{18\pi^2} \log L + \underline{0}(1).$$

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Now let us consider the integral in part b):

$$\frac{1}{9} \int_1^L \int_1^L \frac{\cos \pi(z_1 + z_2)}{\pi(z_1 + z_2)} \cdot \frac{\sin \pi(z_1 - z_2)}{\pi \cdot (z_1 - z_2)}$$

$$\left(\frac{z_1^{\frac{2}{3}} + z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}} + z_2^{\frac{2}{3}}}{z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}}} \right) \cdot \left(\frac{z_1^{\frac{2}{3}} - z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}} + z_2^{\frac{2}{3}}}{z_1^{\frac{1}{3}} \cdot z_2^{\frac{1}{3}}} \right) dz_1 dz_2 =$$

$$= \frac{1}{9} \cdot \int_2^{2L} \frac{\cos \pi u}{\pi u} \cdot \int_0^{u-2} \frac{\sin \pi v}{\pi v} \cdot \left[\left(\frac{u+v}{u-v} \right)^{\frac{2}{3}} + \left(\frac{u-v}{u+v} \right)^{\frac{2}{3}} + 1 \right] dv du.$$

It is not difficult to see that oscillations of trigonometric functions make this integral to be of order of constant. To show this we write the inner integral as $I_4(u) + I_5(u) + I_6(u)$, where

$$I_4(u) = \int_0^{u-2} \frac{\sin \pi v}{\pi v} \cdot \left(\frac{u+v}{u-v} \right)^{\frac{2}{3}} dv,$$

$$I_5(u) = \int_0^{u-2} \frac{\sin \pi v}{\pi v} \cdot \left(\frac{u+v}{u-v} \right)^{\frac{1}{3}} dv,$$

$$I_6(u) = \int_0^{u-2} \frac{\sin \pi v}{\pi v} dv.$$

Integration by parts gives $I_6(u) = \frac{1}{2} + \underline{0}(u^{-1})$. Consider now

$$I_4(u) = \int_0^{u-2u^{\frac{1}{4}}} \left(\frac{\sin \pi v}{\pi v} \right) \cdot \left(\frac{u+v}{u-v} \right)^{\frac{2}{3}} dv + \int_{u-2u^{\frac{1}{4}}}^{u-2} \left(\frac{\sin \pi v}{\pi v} \right) \cdot \left(\frac{u+v}{u-v} \right)^{\frac{2}{3}} dv.$$

$$(3.23)$$

The second integral in (3.23) is $\underline{0}(u^{\frac{1}{4}} \cdot u^{-1} \cdot u^{\frac{2}{3}}) = \underline{0}(u^{-\frac{1}{12}})$. As for the first one we introduce $\chi([v] \text{ even})$, an indicator of the set where the integer part of v is even, and write it in the following form

$$\int_0^{u-2u^{\frac{1}{4}}} \frac{\sin \pi v}{\pi} \cdot \chi([v] \text{ even}) \cdot \left(\frac{1}{v} \cdot \left(\frac{u+v}{u-v} \right)^{\frac{2}{3}} - \frac{1}{v+1} \right.$$

$$\left. \left(\frac{u+v+1}{u-v-1} \right)^{\frac{2}{3}} \right) dv + \underline{0}(u^{-\frac{1}{2}}).$$

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The absolute value of the last expression is estimated from above by

$$\int_0^{u-2u^{\frac{1}{4}}} \frac{|\sin \pi v|}{\pi} \cdot \left(\frac{v \cdot (u-v)^{\frac{2}{3}} - (u-v-1)^{\frac{2}{3}} \cdot (v+1)}{v \cdot (v+1) \cdot (u-v)^{\frac{2}{3}} \cdot (u-v-1)^{\frac{2}{3}}} \right) dv$$

$$= \int_0^{u-2u^{\frac{1}{4}}} \frac{|\sin \pi v|}{\pi} \cdot \left(\frac{(u-v)^{\frac{2}{3}} - (u-v-1)^{\frac{2}{3}}}{(v+1) \cdot (u-v)^{\frac{2}{3}} \cdot (u-v-1)^{\frac{2}{3}}} \right) dv$$

$$+ \int_0^{u-2u^{\frac{1}{4}}} \frac{|\sin \pi v|}{\pi} \cdot \left(\frac{1}{v \cdot (v+1) \cdot (u-v)^{\frac{2}{3}}} \right) dv = \underline{0}(u^{-\frac{1}{6}}).$$

Therefore $I_4(u) = \underline{0}(u^{-\frac{1}{6}})$, and similarly $I_5(u) = \underline{0}(u^{-\frac{1}{12}})$. As a result

$$\frac{1}{9} \int_2^{2L} \frac{\sin \pi u}{\pi u} (I_4(u) + I_5(u) + I_6(u)) du = \frac{1}{18} \int_2^{2L} \frac{\sin \pi u}{\pi u} du$$

$$+ \frac{1}{9} \int_2^{2L} \frac{\sin \pi u}{\pi u} \cdot \underline{0}(u^{-\frac{1}{12}}) du = \underline{0}(1).$$

Lemma 5 is proven. $\qquad\qquad\square$

As a result of the last two lemmas we have

$$\int_{-L}^{-1} \int_{-L}^{-1} S^2(z_1, z_2) dz_1 dz_2 = L - \frac{2}{3\pi^2} \log L + \frac{1}{18\pi^2} \log L + \underline{0}(1) =$$

$$L - \frac{11}{18\pi^2} \log L + \underline{0}(1).$$

Remember that S was defined as $S(z_1, z_2) = Q_{0,0}^{(1)}(z_1, z_2) + Q_{0,0}^{(2)}(z_1, z_2)$. To finish the proof of the CLT for $\#(-T, +\infty)$ we just need to show that the remainder term $U(z_1, z_2) = Q(z_1, z_2) - S(z_1, z_2)$ is insignificant in the following sense:

Lemma 6
  a) $\int_{-L}^{-1} \int_{-L}^{-1} U^2(z_1, z_2) dz_1 dz_2 = \underline{0}(1)$
  b) $\int_{-L}^{-1} \int_{-L}^{-1} U(z_1, z_2) \cdot S(z_1, z_2) dz_1 dz_2 = \underline{0}(1)$
  c) $\int_{-L}^{-1} U(z, z) dz = \underline{0}(1)$

Proof. We shall establish part a). Parts b) and c) can be treated in a similar manner. Repeating the calculations of the last two lemmas, it is easy to see

that for any fixed indices (i, m, n) such that $(i - 1)(i - 2) + m + n > 0$, we have

$$\int_{-L}^{-1} \int_{-L}^{-1} \left(Q_{m,n}^{(i)}(z_1, z_2)\right)^2 dz_1 dz_2 = \underline{0}(1).$$

Let us now choose N to be sufficiently large and write $U(z_1, z_2) = U_N(z_1, z_2) + V_N(z_1, z_2)$, where

$$U_N(z_1, z_2) = \sum_{i=1}^{2} \sum_{0 < m+n \leq N} Q_{m,n}^{(i)}(z_1, z_2) + \sum_{i=3}^{6} \sum_{0 \leq m+n \leq N} Q_{m,n}^{(i)}(z_1, z_2).$$

We see that $\int_{-L}^{-1} \int_{-L}^{-1} (U_N(z_1, z_2))^2 dz_1 dz_2 = \underline{0}(1)$. Asymptotic formulas (3.6)–(3.11) imply that

$$
\begin{aligned}
|V_N(z_1, z_2)| &\leq \frac{\mathrm{const}_N}{|z_1^{\frac{2}{3}} - z_2^{\frac{2}{3}}|} \cdot \left(z_1^{-2N} + z_n^{-2N}\right) \\
&\leq \frac{\mathrm{const}_N}{|z_1 - z_2|} \cdot \frac{3}{2} \cdot \left(z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}\right) \cdot \left(z_1^{-2N} + z_2^{-2N}\right).
\end{aligned}
\tag{3.24}
$$

It follows from (3.24) that if we choose $N \geq 2$ then

$$\iint_{|z_1 - z_2| \geq \frac{1}{z_2^2}} (V_n(z_1, z_2))^2 \, dz_1 dz_2 = \underline{0}(1). \tag{3.25}$$

(The integration in (3.25) is over the subset of $[1, L] \times [1, L]$) . Indeed, to estimate the integral over z_2 we write

$$\int_{|z_1 - z_2| \geq \frac{1}{z_2^2}} \left(\frac{z_1}{(z_1 - z_2)} \cdot \frac{1}{z_2^{2N}}\right)^2 dz_2 \leq \int_1^L \frac{z_2^4}{(z_1 - z_2)^2 + 1} \cdot z_1^{\frac{2}{3}} \cdot \frac{1}{z_2^{4N}} dz_2$$

$$\leq \int_1^L \frac{z_1^{\frac{2}{3}}}{(z_1 - z_2)^2 + 1} \cdot z_2^{-4} dz_2 = \underline{0}(z_1^{-\frac{4}{3}}).$$

Integrating over z_1 we arrive at (3.25). To integrate V_N^2 near the diagonal we observe that kernels $Q(z_1, z_2)$, $Q_{m,n}^{(i)}(z_1, z_2)$ are bounded in $[1, +\infty) \times [1, +\infty)$; therefore there exists some const_N' such that $|V_N(z_1, z_2)| \leq \mathrm{const}_N'$ and $\iint_{|z_1 - z_2| \leq \frac{1}{z_2^2}} (V_N(z_1, z_2))^2 dz_1 dz_2 \leq \int_1^{+\infty} \frac{2 \cdot \mathrm{const}_N'}{z_2^2} dz_2 = \underline{0}(1)$. Lemma 6 is proven. $\quad\square$

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Taking $L = \frac{2}{3\pi}T^{\frac{3}{2}}$ we deduce from Lemmas 4-6 and (3.5) that

$$
\mathrm{Var}\left(\#\left(y_i \in \left(-T, -\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}\right)\right)\right) = \int_{-T}^{-\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}} K(y,y)dy
$$

$$
- \int_{-T}^{-\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}} \int_{-T}^{-\left(\frac{3\pi}{2}\right)^{\frac{2}{3}}} K^2(y_1,y_2)dy_1dy_2
$$

$$
= \frac{2}{3\pi}T^{\frac{2}{3}} + \underline{0}(1) - \frac{2}{3\pi}T^{\frac{2}{3}}
$$

$$
+ \frac{11}{18\pi^2}\log\left(\frac{2}{3\pi}T^{\frac{3}{2}}\right) + \underline{0}(1)
$$

$$
= \frac{11}{12\pi^2}\log T + \underline{0}(1).
$$

This finishes the proof of Proposition 2 as well as the proof of the CLT for $\#(-T, +\infty)$.

In a very similar way one proves the CLT for arbitrary $\nu_k(T) = \#\left((-kT, -(k-1)T)\right)$, $k > 1$. To prove the result for the joint distribution of $\{\nu_k(T)\}$ we note that the decay of $K(x,y)$ off the diagonal implies

$$
C_{1,1}(\nu_k(T),\nu_l(T)) = \mathrm{Cov}\,(\nu_k(T),\nu_l(T)) = \int_{-kT}^{-(k-1)T}\int_{-lT}^{-(l-1)T} K^2(x_1,x_2)dx_1,dx_2 =
$$

$$
- \mathrm{Trace}\chi_{[-lT,-(l-1)T)} \cdot K \cdot \chi_{[-kT,-(k-1)T)} \cdot K = \underline{O}(1) \quad \text{if} \quad |k-l| > 1.
$$

This together with

$$
\mathrm{Var}\left(\sum_{l=1}^{k}\nu_l(T)\right) = 11/(12\pi^2)\log T + \underline{Q}(1), \quad \mathrm{Var}(\nu_k(T)) = 11/(12\pi^2)\log T + \underline{Q}(1), \quad ,k=1,2,\ldots
$$

implies that

$$
C_{1,1}(\nu_k(T),\nu_l(T)) = \mathrm{Cov}\,(\nu_k(T),\nu_l(T)) =
$$

$$
11/(12\pi^2)\log T + \underline{Q}(1)
$$

for $|k-l| = 1$. Therefore as $T \to \infty$.

$$
E\frac{\nu_k(T) - E\nu_k(T)}{\sqrt{\mathrm{Var}\,\nu_k(T)}} \cdot \frac{\nu_l(T) - E\nu_l(T)}{\sqrt{Var\,\nu_l(T)}} \to \delta_{k,l} - 1/2\,\delta_{k,l-1} - 1/2\,\delta_{k,l+1}
$$

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To take care of the joint cumulants of higher order it is enought to prove

Lemma 7 *Let at least two indices in $(k_1, \ldots, \mathcal{K}_s)$ are non-zero. Then*

$$C_{k_1,\ldots,k_s}(\nu_1(T), \ldots, \nu_s(T)) = \underline{Q}(\log T)$$

.

Proof According to Proposition 1 Lemma 7 follows from

Lemma 8

$$\text{Trace} \chi_{l_1} K \chi_{l_2} K \ldots K \chi_{l_s} K \chi_{l_1} = \underline{Q}(\log T)$$

where χ_{l_j} are the indicators of the intervals $(-l_j T, -(l_j - 1)T]$, $l_j \in Z_+^1$ and at least two intervals are disjoint .

Proof This has been already established for $s = 2$. Let $s > 2$. Since not all indices coincide by cyclicity of the trace we may assume $l_1 \neq l_2$. Now if $l_1 = l_3$ we can use the positivity of $\chi_{l_1} K \chi_{l_2} K \chi_{l_1}$ to write

$$|\text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1} K \chi_{l_4} \ldots K \chi_{l_s} K \chi_{l_1})| \leq \text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) \cdot \| K \chi_{l_4} \ldots K \chi_{l_s} K \chi_{l_1} \|$$

$$\leq \text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1})$$

where we used $\| K \| \leq 1$, $\| \chi_{l_j} \| \leq 1$. Since $\text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) = \underline{Q}(\log T)$ Lemma 8 is proven when $l_1 = l_3 \neq l_2$. If $l_1 \neq l_3$ one more trick is needed. Let us denote

$$D_1 = \chi_{l_1} K \chi_{l_2}, \quad D_2 = K \chi_{l_3} K \ldots \chi_{l_s} K \chi_{l_1}.$$

Then

$$\text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_3} \ldots \chi_{l_s} K \chi_{l_1}) = \text{Trace}(D_1 D_2)$$

$$\leq (\text{Trace}(B_1 B_1^*))^{1/2} (\text{Trace}(D_2 D_2^*))^{1/2}$$

(see [RS], volume I, section VI.6). As before

$$\text{Trace}(D_1 D_1^*) = \text{Trace}(\chi_{l_1} K \chi_{l_2} K \chi_{l_1}) = \underline{Q}(\log T)$$

To obtain a similar bound for $\text{Trace}(D_1 D_1^*)$ we define $1 < p \leq s$ as the maximal index such that $l_p \neq l_1$. Since we assume in Lemma 8 that there are at least two different indices, such p always exists. Then

$$\text{Trace}\left(K \chi_{l_3} K \ldots \chi_{l_p} K \chi_{l_1} \ldots K \chi_{l_1} \ldots K \chi_{l_1}\right) \cdot \left(K \chi_{l_3} K \ldots \chi_{l_p} K \chi_{l_1} \ldots K \chi_{l_1} \ldots K \chi_{l_1}\right)^* =$$

$$\text{Trace}\left(K\chi_{l_3}K\ldots\chi_{l_p}K\chi_{l_1}\ldots K\chi_{l_1}\ldots K\chi_{l_1}\right)\cdot\left(\chi_{l_1}K\ldots\chi_{l_1}K\ldots K\chi_{l_1}K\chi_{l_p}\ldots K\chi_{l_3}K\right).$$

Using the identity $\quad \text{Trace}(D_1 D_2) = \text{Trace}(D_2 D_1)\quad$ where

$$D_1 = K\chi_{l_3}K\ldots\chi_{l_{p-1}}K,\ \ D_2 = \chi_{l_p}K\chi_{l_1}K\ldots\chi_{l_1}K\chi_{l_1}\ldots\chi_{l_1}K\chi_{l_p}K\chi_{l_{p-1}}K\ldots K\chi_{l_3}K,$$

we can rewrite and estimate the r.h.s. as

$$\left|\text{Trace}\left(\chi_{l_p}K\chi_{l_1}\ldots\chi_{l_1}K\ldots\chi_{l_1}K\chi_{l_p}\right)\cdot\left(K\chi_{l_{p-1}}\ldots K\chi_{l_3}KK\chi_{l_3}K\ldots\chi_{l_{p-1}}K\right)\right|$$

$$\leq \text{Trace}\left(\chi_{l_p}K\chi_{l_1}\ldots K\chi_{l_1}K\ldots\chi_{l_1}K\chi_{l_p}\right)\cdot\parallel K\chi_{l_{p-1}}\ldots K\chi_{l_3}KK\chi_{l_3}K\ldots\chi_{l_{p-1}}K\parallel$$

Here we used the positivity of $\quad \chi_{l_p}K\chi_{l_1}K\chi_{l_1}\ldots K\chi_{l_1}\ldots K\chi_{l_1}\quad$. The norm
of the last factor is again not greater than 1. Finally,

$$\text{Trace}\left(\chi_{l_p}K\chi_{l_1}K\chi_{l_1}\ldots K\chi_{l_1}\ldots\chi_{l_1}K\chi_{l_p}\right) = \text{Trace}\left(\chi_{l_1}K\chi_{l_p}K\chi_{l_1}\ldots K\chi_{l_1}\ldots K\chi_{l_1}\right)\leq$$

$$\text{Trace}\left(\chi_{l_1}K\chi_{l_p}K\chi_{l_1}\right)\cdot 1 = \underline{O}(\log T).$$

Here we also used cyclicity of the trace. Combining the estimates for $\text{Trace}D_1 D_1^*$
and $\text{Trace}D_1 D_1^*$ we finish the proof of the Lemmas 7 and 8. $\qquad\square$
It follows from Lemma 7 that the higher joint cumulants of the normalized
random variables go to zero which implies that the limiting distribution function is gaussian with the known covariance function. Theorem 1 is proven
. $\qquad\square$

4 Proof of Theorem 2 and Similar Results for the Classical Compact Groups

The Bessel kernel has the form (see §1)

$$K(y_1, y_2) = \frac{J_\alpha(\sqrt{y_1})\cdot\sqrt{y_2}\cdot J_\alpha'(\sqrt{y_2}) - \sqrt{y_1}\cdot J_\alpha'(\sqrt{y_1})\cdot J_\alpha(\sqrt{y_2})}{2(y_1 - y_2)}, \qquad (4.1)$$

$$y_1, y_2 \in (0, +\infty),\ \alpha > -1.$$

The level density is given by

$$\rho_1(y) = K(y, y) = \frac{1}{4}J_\alpha(\sqrt{y})^2 - \frac{1}{4}J_{\alpha+1}(\sqrt{y})\cdot J_{\alpha-1}(\sqrt{y}). \qquad (4.2)$$

The asymptotic formula for large y is well known in the case of Bessel functions (see asymptotic expansion in (4.5) below). In particular, one can see that

$$\rho_1(y) \sim \frac{1}{2\pi\sqrt{y_1}} \text{ for } y \to +\infty. \tag{4.3}$$

The last formula suggests to make the (unfolding) change of variables $z_i = \frac{\sqrt{y_i}}{\pi}$, $i = 1, 2$. The kernel Q corresponding to the new evenly spaced random point field is given by

$$
\begin{aligned}
Q(z_1, z_2) &= 2\pi y_1^{\frac{1}{4}} \cdot y_2^{\frac{1}{4}} \cdot K(y_1, y_2) \\
&= z_1^{\frac{1}{2}} z_2^{\frac{1}{2}} \cdot \frac{J_\alpha(\pi z_1) \cdot \pi z_2 \cdot J_\alpha'(\pi z_2) - \pi z_1 \cdot J_\alpha'(\pi z_1) \cdot J_\alpha(\pi z_2)}{z_1^2 - z_2^2}.
\end{aligned}
\tag{4.4}
$$

The asymptotic expansion of the Bessel function at infinity is due to Hankel (see, for example, [Ol]).

$$
\begin{aligned}
J_\alpha(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot \Big[& \cos\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty}(-1)^s \cdot \frac{A_{2s}(\alpha)}{z^{2s}} \\
& - \sin\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty}(-1)^s \cdot \frac{A_{2s+1}(\alpha)}{z^{2s+1}} \Big],
\end{aligned}
\tag{4.5}
$$

where

$$A_0(\alpha) = 1, \ A_s(\alpha) = \frac{(4\alpha^2 - 1^2) \cdot (4\alpha^2 - 3^2) \dots (4\alpha^2 - (2s-1)^2)}{s! \cdot 8^s}. \tag{4.6}$$

Similarly,

$$
\begin{aligned}
J_\alpha'(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cdot \Big[& -\sin\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty}(-1)^s \cdot \frac{B_{2s}(\alpha)}{z^{2s}} \\
& - \cos\left(z - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi\right) \cdot \sum_{s=0}^{\infty}(-1)^s \cdot \frac{B_{2s+1}(\alpha)}{z^{2s+1}} \Big].
\end{aligned}
\tag{4.7}
$$

The coefficients $B_s(\alpha)$ can be obtained from (4.5)–(4.6); for example, $B_0(\alpha) = 1$.

It follows from (4.5)–(4.7) that for $\alpha = \pm\frac{1}{2}$

$$Q(z_1, z_2) = \frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} + \frac{\sin \pi(z_1 + z_2 - \alpha - \frac{1}{2})}{\pi(z_1 + z_2)}, \qquad (4.8)$$

which can be further simplified as

$$\frac{\sin \pi(z_1 - z_2)}{\pi(z_1 - z_2)} \mp \frac{\sin \pi(z_1 + z_2)}{\pi(z_1 + z_2)}.$$

For general values of α a small remainder term appears at the r.h.s. of (4.8). To write the asymptotic expansion, we represent $Q(z_1, z_2)$ as the sum of six kernels: $Q(z_1, z_2) = \sum\limits_{i=1}^{6} Q^{(i)}(z_1, z_2)$, where

$$Q^{(1)}(z_1, z_2) \sim \frac{\sin \pi(z_1 - z_2)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n,m=0}^{\infty} (-1)^{n+m} \cdot A_{2n}(\alpha) B_{2m}(\alpha)$$
$$n \qquad \cdot \left(z_1^{1-2n} \cdot z_2^{-2m} + z_1^{-2m} \cdot z_2^{1-2n} \right), \qquad (4.9)$$

$$Q^{(2)}(z_1, z_2) \sim \frac{\sin \pi(z_1 + z_2 - \alpha - \frac{1}{2})}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n,m=0}^{\infty} (-1)^{n+m} \cdot A_{2n}(\alpha)$$
$$\cdot B_{2m}(\alpha) \cdot \left(z_1^{1-2n} \cdot z_2^{-2m} - z_1^{-2m} \cdot z_2^{1-2n} \right), \qquad (4.10)$$

$$Q^{(3)}(z_1, z_2) \sim \frac{2\cos\left(\pi z_1 - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) \cdot \cos\left(\pi z_2 - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right)}{\pi(z_1^2 - z_2^2)}$$
$$\cdot \sum_{n,m=0}^{\infty} (-1)^{n+m+1} \cdot A_{2n}(\alpha) \cdot B_{2m+1}(\alpha) \cdot \left(z_1^{-2n} \cdot z_2^{-2m} + z_1^{-2m} \cdot z_2^{-2n} \right),$$
$$(4.11)$$

$$Q^{(4)}(z_1, z_2) \sim \frac{2\sin\left(\pi z_1 - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right) \cdot \sin\left(\pi z_2 - \frac{\pi\alpha}{2} - \frac{\pi}{4}\right)}{\pi(z_1^2 - z_2^2)}$$
$$\cdot \sum_{n,m=0}^{\infty} (-1)^{n+m} \cdot A_{2n+1}(\alpha) \cdot B_{2m}(\alpha) \cdot \left(z_1^{-1-2n} \cdot z_2^{1-2m} + z_1^{1-2m} \cdot z_2^{-1-2n} \right),$$
$$(4.12)$$

$$Q^{(5)}(z_1, z_2) \sim \frac{\sin \pi (z_1 - z_2)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n,m=0}^{\infty} (-1)^{n+m+1} \cdot A_{2n+1}(\alpha) \cdot B_{2m+1}(\alpha)$$
$$\cdot \left(z_1^{-1-2n} \cdot z_2^{-2m} + z_1^{-2m} \cdot z_2^{-1-2n} \right), \tag{4.13}$$

$$Q^{(6)}(z_1, z_2) \sim \frac{\sin \pi \left(z_1 + z_2 - \alpha - \frac{1}{2} \right)}{\pi(z_1^2 - z_2^2)} \cdot \sum_{n,m=0}^{\infty} (-1)^{n+m+1} \cdot A_{2n+1}(\alpha)$$
$$\cdot B_{2m+1}(\alpha) \cdot \left(z_1^{-1-2n} \cdot z_2^{-2m} - z_1^{-2m} \cdot z_2^{-1-2n} \right). \tag{4.14}$$

The analysis of (4.9)–(4.14) is very similar to §3. One can see that the only contribution to the leading term of the variance comes from $Q_{0,0}^{(1)}(z_1, z_2) + Q_{0,0}^{(2)}(z_1, z_2)$ which is exactly the r.h.s. of (4.8). It can be shown by a straightforward calculation that

$$\int_0^L \left(Q_{0,0}^{(1)}(z, z) + Q_{0,0}^{(2)}(z, z) \right) dz \sim L + \underline{0}(1),$$

$$\int_0^L \int_0^L \left(Q_{0,0}^{(1)}(z_1, z_2) + Q_{0,0}^{(2)}(z_1, z_2) \right)^2 dz_1 dz_2 \sim L - \frac{1}{2\pi^2} \log L + \underline{0}(1).$$

Taking into account that $L = \frac{T^{\frac{1}{2}}}{\pi}$ we finish the proof. $\qquad\square$

The kernels $\frac{\sin \pi(x-y)}{\pi(x-y)} \pm \frac{\sin \pi(x+y)}{\pi(x+y)}$ are well known in Random Matrix Theory. For one, they are the kernels of restrictions of the sine-kernel integral operator to the subspaces of even and odd functions and play an important role in spacings distribution in G.O.E. and G.S.E. ([Me]). They also appear as the kernels of limiting correlation functions in orthogonal and symplectic groups near $\lambda = 1$ ([So1]). Let us start with the even case. Consider the normalized Haar measure on $SO(2n)$. The eigenvalues of matrix M can be arranged in pairs:

$$\exp(i\theta_1), \exp(-i\theta_1), \ldots, \exp(i\theta_n), \exp(-i\theta_n), 0 \le \theta_1, \theta_2, \ldots, \theta_n \le \pi. \tag{4.15}$$

In the rescaled coordinates near the origin $x_i = (2n-1) \cdot \frac{\theta_i}{2\pi}$, $i = 1, \ldots, n$, the k-point correlation functions are equal to

$$R_{n,k}(x_1, \ldots, x_k) = \det \left(\frac{\sin \pi(x_i - x_j)}{(2n-1) \cdot \sin(\pi \cdot (x_i - x_j)/(2n-1))} \right.$$
$$\left. + \frac{\sin \pi(x)i + x_j)}{(2n-1) \cdot \sin(\pi(x_i + x_j)/(2n-1))} \right)_{i,j=1,\ldots k}. \tag{4.16}$$

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In the limit $n \to \infty$ the kernel in (4.16) becomes $\frac{\sin \pi (x_i - x_j)}{\pi (x_i - x_j)} + \frac{\sin \pi (x_i + x_j)}{\pi (x_i + x_j)}$. If we consider rescaling near arbitrary $0 < \theta < \pi$ the limiting kernel will be just the sine kernel.

Let us now consider the $SO(2n+1)$ case. The first $2n$ eigenvalues of $M \in SO(2n+1)$ can be arranged in pairs as in (4.15). The last one equals 1. In the rescaled coordinates near $\theta = 0$

$$x_i = \frac{n\theta_i}{\pi}, \quad i = 1, \ldots, n,$$

the k-point correlation functions are given by the formula

$$
\begin{aligned}
R_{n,k}(x_1, \ldots, x_k) = \det\bigg(& \frac{\sin \pi (x_i - x_j)}{2n \cdot \sin(\pi \cdot (x_i - x_j)/2n)} \\
& - \frac{\sin \pi (x_i + x_j)}{2n \cdot \sin(\pi \cdot (x_i + x_j)/2n)} \bigg)_{i,j=1\ldots k}.
\end{aligned}
\tag{4.17}
$$

In the limit $n \to \infty$ the kernel in (4.17) becomes $\frac{\sin \pi (x_i - x_j)}{\pi \cdot (x_i - x_j)} - \frac{\sin \pi (x_i + x_j)}{\pi \cdot (x_i + x_j)}$. If we again consider rescaling near $0 < \theta < \pi$, the limiting kernel appears to be the sine kernel. The case of symplectic group is very similar. Let $M \in Sp(n)$. Rescaled k-point correlation functions are given by

$$
\begin{aligned}
R_{n,k}(x_1, \ldots, x_k) = \det\bigg(& \frac{\sin \pi (x_i - x_j)}{(2n+1) \cdot \sin(\pi(x_i - x_j)/(2n+1)} \\
& - \frac{\sin \pi (x_i + x_j)}{(2n+1) \cdot \sin(\pi(x_i + x_j)/(2n+1))} \bigg)_{i,j=1,\ldots k}.
\end{aligned}
\tag{4.18}
$$

One can then deduce the following result from the Costin-Lebowitz Theorem.

Theorem 3 *Consider the normalized Haar measure on $SO(n)$ or $Sp(n)$. Let $\theta \in [0, \pi)$, δ_n be such that $0 < \delta_n < \pi - \theta - \epsilon$ for some $\epsilon > 0$ and $n \cdot \delta_n \to +\infty$. Denote by ν_n the number of eigenvalues in $[\theta, \theta + \delta_n]$. We have $E\nu_n = \frac{n}{\pi} \cdot \delta_n + \underline{0}(1)$,*

$$
\operatorname{Var} \nu_n = \begin{cases} \frac{1}{\pi^2} \log(n \cdot \delta_n) + \underline{0}(1) & \text{if } \theta > 0, \\ \frac{1}{2\pi^2} \log(n \cdot \delta_n) + \underline{0}(1) & \text{if } \theta = 0 \end{cases}
$$

and the normalized random variable $\frac{\nu_n - E\nu_n}{\sqrt{\operatorname{Var} \nu_n}}$ converges in distribution to the normal law $N(0,1)$.

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Proof. Let $K_n(x, y)$ be the kernel in (4.16), (4.17), or (4.18). We observe that $0 \leq K_n \cdot \xi_I \leq Id$ as a composition of projection, Fourier transform, another projection and inverse Fourier transform. To check the asymptotics of Var ν_n is an excercise which is left to the reader. $\qquad \square$

The case of several intervals is treated in a similar fasion.

Theorem 4 *Let $\delta_n > 0$ be such that $\delta_n \to 0$, $n\delta_n \to \infty$ and $\nu_{k,n} = \#\left((k-1)\delta_n, k\delta_n]\right)$. Then a sequence of normalized random variables $\frac{\nu_{k,n} - E\nu_{k,n}}{\sqrt{\text{Var } \nu_{k,n}}}$ converges in distribution to the centalized gaussian sequence $\{\xi_k\}_{k=1}^{\infty}\}$ with the covariance function*

$$E\xi_k\xi_l = \begin{cases} \delta_{k,l} - 1/2 \; \delta_{k,l+1} - 1/2 \; \delta_{k,l-1}, & \text{if } k > 0, \; l > 0 \\ \delta_{0,l} - 1/\sqrt{2} \; \delta_{1,l}, & \text{if } \theta = 0 \end{cases}$$

.

Finally we discuss the unitary group $U(n)$.
The eigenvalues of matrix M can be written as:

$$\exp(i\theta_1), \exp(i\theta_2), \ldots, \exp(i\theta_n).d, 0 \leq \theta_1, \theta_2, \ldots, \theta_n \leq 2\pi. \qquad (4.19)$$

In the rescaled coordinates $x_i = n \cdot \frac{\theta_i}{2\pi}$, $i = 1, \ldots, n$, the k-point correlation functions are equal to

$$R_{n,k}(x_1, \ldots, x_k) = \det\left(\frac{\sin \pi(x_i - x_j)}{n \cdot \sin(\pi \cdot (x_i - x_j)/n)}\right)_{i,j=1,\ldots k}. \qquad (4.20)$$

In the limit $n \to \infty$ the kernel in (4.20) becomes the sine kernel. We finish with the analoques of the last two theorems for $U(n)$.

Theorem 5 *Consider the normalized Haar measure on $U(n)$. Let $\theta \in [0, 2\pi)$, δ_n be such that $0 < \delta_n < 2\pi - \theta - \epsilon$ for some $\epsilon > 0$ and $n \cdot \delta_n \to +\infty$. Denote by ν_n the number of eigenvalues in $[\theta, \theta + \delta_n]$. We have $E\nu_n = \frac{n}{2\pi} \cdot \delta_n$,*

$$\text{Var } \nu_n = \frac{1}{\pi^2} \log(n \cdot \delta_n) + \underline{0}(1)$$

and the normalized random variable $\frac{\nu_n - E\nu_n}{\sqrt{\text{Var } \nu_n}}$ converges in distribution to the normal law $N(0, 1)$.

Theorem 6 *Let $\delta_n > 0$ be such that $\delta_n \to 0$, $n\delta_n \to \infty$ and $\nu_{k,n} = \#\left((k-1)\delta_n, k\delta_n]\right)$. Then a sequence of normalized random variables $\frac{\nu_{k,n} - E\nu_{k,n}}{\sqrt{\text{Var } \nu_{k,n}}}$*

converges in distribution to the centalized gaussian sequence $\{\xi_k\}_{k=1}^\infty\}$ with the covariance function

$$E\xi_k\xi_l = \delta_{k,l} - 1/2 \ \delta_{k,l+1} - 1/2 \ \delta_{k,l-1}$$

. Remark 5. Results similar to Theorems 5,6 in the regime $\delta_n = \delta > 0$ have been also established by K.Wieand ([W]).

Remark 6. For the results about smooth linear statistics in the Classical Compact Groups we refer the reader to [DS], [Jo1], [So3].

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